Algebra Colloquium **15** : 2 (2008) 207–222

Algebra Colloquium

On the Yoneda Ext-Algebras of Semiperfect Algebras*

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Received 10 December 2005
Revised 14 May 2007
Communicated by Quanshui Wu

Abstract. It is proved that the Yoneda Ext-algebras of Morita equivalent semiperfect algebras are graded equivalent. The Yoneda Ext-algebras of noetherian semiperfect algebras are studied in detail. Let A be a noetherian semiperfect algebra with Jacobson radical J. We construct a right ideal $\overline{E}(A)$ of the Yoneda algebra $E(A) = \mathrm{Ext}_A^*(A/J, A/J)$, which plays an important role in the discussion of the structure of $E(A)$. An extra grading is introduced to $\overline{E}(A)$, by which we give a description of the right ideal of $E(A)$ generated by $\mathrm{Ext}_A^1(A/J, A/J)$, and we give a necessary and sufficient condition for a notherian semiperfect algebra to be higher quasi-Koszul. Finally, it is shown that the quasi-Koszulity of a noetherian semiperfect algebra is a Morita invariant.

2000 Mathematics Subject Classification: 16W30, 16E40

Keywords: Yoneda Ext-algebra, semiperfect algebra, (higher) quasi-Koszul algebra

The Hochschild cohomology algebras of non-commutative algebras are extensively studied. Let A and B be two algebras. It is well known that if the derived categories $D^b(A)$ and $D^b(B)$ are equivalent, then the Hochschild cohomology algebras $HH(A)$ and $HH(B)$ are isomorphic (see [7]). In particular, if A and B are Morita equivalent, then $HH(A) \cong HH(B)$. However, the structure of the Yoneda Ext-algebras of non-commutative algebras is not very clear. A natural question is: if the algebras A and B are Morita equivalent, are the Yoneda Ext-algebras $E(A)$ and $E(B)$ Morita equivalent? The answer is affirmative if A and B are semiperfect. In fact, we prove that $E(A)$ and $E(B)$ are graded equivalent.

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*The second author is supported by NSFC (No. 10501041).

The concept of Koszul algebras was introduced by Priddy [9] in 1970. Usually, a Koszul algebra is a graded algebra. Koszul algebras have played an important role in commutative algebra, algebraic topology, Lie algebra and quantum groups. Applications of Koszul algebras and examples from different fields can be found in [2]. In 1996, Green and Martínez Villa generalized the concept of Koszul algebras to the non-graded case (see [5] and also [6]). A *quasi-Koszul algebra* is a noetherian semiperfect algebra with certain homology properties. As an example, Green and Martínez Villa showed in [5] that the Auslander algebra of a finite dimensional algebra of finite type over an algebraically closed field is a quasi-Koszul algebra. In fact, ample examples can be found in commutative algebra. For instance, regular local algebras are quasi-Koszul algebras. We generalize the concept of quasi-Koszul algebras to higher quasi-Koszul algebras. The Ext-algebras of (higher) quasi-Koszul algebras are studied in detail. The structure of Ext-algebras of higher quasi-Koszul algebras is much more complicated than that of quasi-Koszul algebras since the minimal resolution of the trivial module of a higher quasi-Koszul algebra is much more complicated.

Let A be a semiperfect algebra and J the Jacobson radical of A. We construct a graded right ideal $\overline{E}(A)$ of the Yoneda algebra $E(A) = \mathrm{Ext}_A^*(A/J, A/J)$, which plays an important role in the study of the structure of $E(A)$. We introduce an extra grading on $\overline{E}(A)$ so that $\overline{E}(A)$ is a bigraded vector space. The extra grading provides some information of the Yoneda product on $E(A)$. With the help of the extra grading on $\overline{E}(A)$, we give a criterion of a noetherian semiperfect algebra to be a higher quasi-Koszul algebra. Although we cannot describe clearly the subalgebra of $E(A)$ generated by $\mathrm{Ext}_A^1(A/J, A/J)$ as it has been done in [8], we give a description of the right ideal of $E(A)$ generated by $\mathrm{Ext}_A^1(A/J, A/J)$ by the extra grading on $\overline{E}(A)$. We then give a short proof of [5, Theorem 4.4] about quasi-Koszul algebras. We also give a necessary and sufficient condition for a noetherian semiperfect algebra to be a higher quasi-Koszul algebra. Finally, as an application, we prove that the Koszulity of noetherian semiperfect algebras is a Morita invariant.

Throughout this paper, k is a field. All the algebras and modules involved are over the field k.

1 Morita Equivalent Semiperfect Rings

In this section, we show that the Yoneda Ext-algebras of Morita equivalent semiperfect algebras are graded equivalent. We recall some terminologies and notations involved in the discussion of this section.

Let A be an algebra. We use A-Mod to denote the category of all left A-modules. Let Y be a left A-module. Then $\mathrm{add}(Y)$ is the full subcategory of A-Mod consisting of all the direct summands of finite direct sums of copies of Y. Two algebras A and A' are said to be Morita equivalent if there are additive covariant functors $F : A\text{-Mod} \to A'\text{-Mod}$ and $G : A'\text{-Mod} \to A\text{-Mod}$ such that $G \circ F \cong 1_{A\text{-Mod}}$ and $F \circ G \cong 1_{A'\text{-Mod}}$. Let ${}_AX$ be a left A-module. Endowed with the Yoneda product, $\mathrm{Ext}_A^*(X, X)$ is a positively $\mathbb{Z}$-graded algebra, denoted by $\mathcal{E}(X)$. The Ext-group $\mathrm{Ext}_A^*(Y, X)$ is a left $\mathbb{Z}$-graded $\mathcal{E}(X)$-module.

Let E be a $\mathbb{Z}$-graded algebra. We use E-GrMod to denote the category of graded left E-modules. The *shift functor* $S : E\text{-GrMod} \to E\text{-GrMod}$ is an automorphism of E-GrMod defined on objects by $S(X)_n = X_{n+1}$, and in an obvious way on morphisms. Let E' be another $\mathbb{Z}$-graded algebra. An additive covariant functor $V : E\text{-GrMod} \to E'\text{-GrMod}$ is called a *graded functor* if V commutes with the shift functor S and its inverse S^{-1}. A graded functor $V : E\text{-GrMod} \to E'\text{-GrMod}$ is said to be a *graded equivalence* (see [3]) if there is a graded functor $U : E'\text{-GrMod} \to E\text{-GrMod}$ such that $U \circ V \cong 1_{E\text{-GrMod}}$ and $V \circ U \cong 1_{E'\text{-GrMod}}$. Two $\mathbb{Z}$-graded algebras E and E' are said to be *graded equivalent* if there is a graded equivalence $V : E\text{-GrMod} \to E'\text{-GrMod}$. It is shown in [3] that graded equivalent graded algebras are Morita equivalent, but Morita equivalent graded algebras are not necessarily graded equivalent.

The following lemma is obvious. For explicit narrative, we give a proof.

Lemma 1.1. *Let A be an algebra, and let X and Z be left A-modules. Assume $X = Y \oplus Y'$, and let $e \in \mathrm{End}(_AX)$ be the idempotent corresponding to the direct summand Y. Then $\mathrm{Ext}_A^*(Y,Z) \cong \mathrm{Ext}_A^*(X,Z) \cdot e$ as graded left $\mathcal{E}(Z)$-modules. Similarly, $\mathrm{Ext}_A^*(Z,Y) \cong e \cdot \mathrm{Ext}_A^*(Z,X)$ as graded right $\mathcal{E}(Z)$-modules.*

Proof. Choose projective resolutions of Y and Y':

$$\cdots \to P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} Y \to 0,$$

$$\cdots \to P'_n \xrightarrow{d'_n} \cdots \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} Y' \to 0.$$

Let $K_n = \mathrm{Ker}(d_{n-1})$ and $K'_n = \mathrm{Ker}(d'_{n-1})$. Consider the diagram

$$\begin{array}{ccccccccc}
0 \to & K_n \oplus K'_n & \xrightarrow{(\tau_n,\tau'_n)} & P_{n-1} \oplus P'_{n-1} & \to \cdots \to & P_0 \oplus P'_0 & \to & X & \to 0 \\
& {\scriptstyle j_n}\uparrow\downarrow{\scriptstyle \pi_n} & & {\scriptstyle j_{n-1}}\uparrow\downarrow{\scriptstyle \pi_{n-1}} & & {\scriptstyle j_0}\uparrow\downarrow{\scriptstyle \pi_0} & & {\scriptstyle j}\uparrow\downarrow{\scriptstyle \pi} & \\
0 \to & K_n & \xrightarrow{\tau_n} & P_{n-1} & \to \cdots \to & P_0 & \to & Y & \to 0
\end{array}$$

where τ_n and τ'_n are the inclusions, π_i the natural projection, and j_i the natural injection. We have $\mathrm{Ext}_A^n(Y,Z) = \mathrm{Hom}_A(K_n,Z)/\mathrm{Im}(\tau_n^*)$, and $\mathrm{Ext}_A^n(X,Z) = \mathrm{Hom}_A(K_n \oplus K'_n, Z)/\mathrm{Im}(\tau_n^*, \tau_n'^*)$. For $g \in \mathrm{Hom}_A(K_n,Z)$, we use $\overline{g}$ to denote its image in $\mathrm{Ext}_A^n(Y,Z)$. Similarly, for $h \in \mathrm{Hom}_A(K_n \oplus K'_n, Z)$, we use $\overline{h}$ to denote its image in $\mathrm{Ext}_A^n(X,Z)$. Define $\phi : \mathrm{Ext}_A^*(Y,Z) \to \mathrm{Ext}_A^*(X,Z) \cdot e$ by $\phi(\overline{g}) = \overline{g \circ \pi_n} \cdot e \in \mathrm{Ext}_A^*(X,Z) \cdot e$. It is not hard to check that ϕ is a left graded $\mathcal{E}(Z)$-module morphism. On the other hand, we define $\varphi : \mathrm{Ext}_A^*(X,Z) \cdot e \to \mathrm{Ext}_A^*(Y,Z)$ by $\varphi(\overline{h} \cdot e) = \overline{h \circ j_n}$. Now we have $\phi\varphi(\overline{h} \cdot e) = \phi(\overline{h \circ j_n}) = \overline{h \circ j_n \circ \pi_n} \cdot e = \overline{h} \cdot e$ by the definition of Yoneda product, and $\varphi\phi(\overline{g}) = \varphi(\overline{g \circ \pi_n} \cdot e) = \overline{g \circ \pi_n \circ j_n} = \overline{g}$. Hence, we have $\mathrm{Ext}_A^*(Y,Z) \cong \mathrm{Ext}_A^*(X,Z) \cdot e$ as graded left $\mathcal{E}(Z)$-modules.

Similarly, $\mathrm{Ext}_A^*(Z,Y) \cong e \cdot \mathrm{Ext}_A^*(Z,X)$ as graded right $\mathcal{E}(Z)$-modules. □

Lemma 1.2. *With the notations in Lemma 1.1, we have $\mathcal{E}(Y) \cong e \cdot \mathcal{E}(X) \cdot e$ as graded algebras.*

Proof. With the same notations as in the proof of Lemma 1.1, we have $\mathcal{E}^n(Y) = \mathrm{Ext}_A^n(Y,Y) = \mathrm{Hom}_A(K_n,Y)/\mathrm{Im}(\tau_n^*)$ and $\mathcal{E}^n(X) = \mathrm{Hom}_A(K_n \oplus K'_n, X)/\mathrm{Im}(\tau_n^*, \tau_n'^*)$.

Define $\theta : \mathcal{E}(Y) \to e\cdot\mathcal{E}(X)\cdot e$ by $\theta(\overline{g}) = e\cdot\overline{j \circ g \circ \pi_n}\cdot e$ for $g \in \mathrm{Hom}_A(K_n, Y)$. It is not hard to see that θ is well defined. On the other hand, define $\psi : e \cdot \mathcal{E}(X) \cdot e \to \mathcal{E}(Y)$ by $\psi(e \cdot \overline{f} \cdot e) = \overline{\pi \circ f \circ j_n} \in \mathcal{E}(Y)$ for $f \in \mathrm{Hom}_A(K_n \oplus K'_n, X)$.

We next show that ψ is well defined. Suppose $e \cdot \overline{f} \cdot e = 0$. Consider the diagram

$$\begin{array}{cccccccccc}
0 \to & K_n \oplus K'_n & \xrightarrow{(\tau_n,\tau'_n)} & P_{n-1} \oplus P'_{n-1} & \to \cdots \to & P_0 \oplus P'_0 & \to & X & \to 0 \\
& \downarrow_{j_n \circ \pi_n} & & \downarrow_{j_{n-1} \circ \pi_{n-1}} & & \downarrow_{j_0 \circ \pi_0} & & \downarrow_{e} & \\
0 \to & K_n \oplus K'_n & \xrightarrow{(\tau_n,\tau'_n)} & P_{n-1} \oplus P'_{n-1} & \to \cdots \to & P_0 \oplus P'_0 & \to & X & \to 0 \\
& \downarrow_{f} & & & & & & & \\
& X & & & & & & &
\end{array}$$

By the definition of Yoneda product, we have $\overline{e \circ f \circ j_n \circ \pi_n} = e \cdot \overline{f} \cdot e = 0$. Hence, $e \circ f \circ j_n \circ \pi_n \in \mathrm{Im}(\tau^*_n, \tau'^*_n)$. So there is a morphism $h : P_{n-1} \oplus P'_{n-1} \to X$ such that $e \circ f \circ j_n \circ \pi_n = h \circ (\tau_n, \tau'_n)$. Clearly, $j_n \circ \pi_n$ is an idempotent. Then we have $e \circ f \circ j_n \circ \pi_n = h \circ (\tau_n, \tau'_n) \circ j_n \circ \pi_n = h \circ (\tau_n, 0) \circ j_n \circ \pi_n$. Assume $h = (h_1, h_2)$ with $h_1 : P_{n-1} \to X$ and $h_2 : P'_{n-1} \to X$. Now $e \circ f \circ j_n \circ \pi_n = h \circ (\tau_n, 0) \circ j_n \circ \pi_n = (h_1 \circ \tau_n, 0) \circ j_n \circ \pi_n$. Thus, $e \circ f \circ j_n = (h_1 \circ \tau_n, 0) \circ j_n$. Since $X = Y \oplus Y'$, we may write $h_1 = (h_{11}, h_{12})$ with $h_{11} : P_{n-1} \to Y$ and $h_{12} : P_{n-1} \to Y'$. Evidently, $\pi \circ e = \pi$ and $\pi \circ h_1 = h_{11}$. We have $\pi \circ f \circ j_n = \pi \circ e \circ f \circ j_n = \pi \circ (h_1 \circ \tau_n, 0) \circ j_n = h_{11} \circ \tau_n$. Hence, $\psi(e \cdot \overline{f} \cdot e) = \overline{\pi \circ f \circ j_n} = \overline{\pi \circ e \circ f \circ j_n} = \overline{h_{11} \circ \tau_n} = 0$. Thus, ψ is well defined.

Since $\psi\theta(\overline{g}) = \psi(e \cdot \overline{j \circ g \circ \pi_n} \cdot e) = \overline{\pi \circ j \circ g \circ \pi_n \circ j_n} = \overline{g}$ and

$$\begin{aligned}
&\theta\psi(e \circ \overline{f} \cdot e) = \theta(\overline{\pi \circ f \circ j_n}) = e \cdot \overline{j \circ \pi \circ f \circ j_n \circ \pi_n} \cdot e \\
= \ &\overline{e \circ j \circ \pi \circ f \circ (j_n \circ \pi_n)^2} = \overline{e \circ f \circ j_n \circ \pi_n} = e \cdot \overline{f} \cdot e,
\end{aligned}$$

we have $\theta\varphi = 1_{e\cdot\mathcal{E}(X)\cdot e}$ and $\psi\theta = 1_{\mathcal{E}(Y)}$.

What remains to show is that θ is a graded algebra morphism.

For $g \in \mathrm{Hom}_A(K_n, Y)$ and $h \in \mathrm{Hom}_A(K_m, Y)$, we have $\overline{g} \in \mathcal{E}^n(Y)$ and $\overline{h} \in \mathcal{E}^m(Y)$. Consider the diagram

$$\begin{array}{cccccccccc}
0 \to & K_{n+m} & \xrightarrow{\tau_{n+m}} & P_{n+m-1} & \to \cdots \to & P_n \to & K_n & \to & P_{n-1} & \to \cdots \\
& \downarrow_{g_m} & & \downarrow_{g_{m-1}} & & \downarrow_{g_0} & \downarrow_{g} & & & \\
0 \to & K_m & \xrightarrow{\tau_m} & P_{m-1} & \to \cdots \to & P_0 \to & Y & \to & 0 & \\
& \downarrow_{h} & & & & & & & & \\
& Y & & & & & & & &
\end{array}$$

where $g_0, \ldots, g_m$ are induced by g. Then $\overline{h} \cdot \overline{g} = \overline{h \circ g_m}$. So $\theta(\overline{h} \cdot \overline{g}) = \theta(\overline{h \circ g_m}) = e \cdot \overline{j \circ h \circ g_m \circ \pi_{n+m}} \cdot e$. On the other hand, $\theta(\overline{h}) \cdot \theta(\overline{g}) = e \cdot \overline{j \circ h \circ \pi_m} \cdot e \cdot \overline{j \circ g \circ \pi_n} \cdot e$. Consider the following commutative diagram

$$\begin{array}{cccccccc}
0 \to & K_{n+m} \oplus K'_{n+m} & \to & P_{n+m-1} \oplus P'_{n+m-1} & \to \cdots \to & P_n \oplus P'_n \to & K_n \oplus K'_n & \to \cdots \\
& \downarrow_{(g_m, 0)} & & \downarrow_{(g_{m-1}, 0)} & & \downarrow_{(g_0, 0)} & \downarrow_{j \circ g \circ \pi_n} & \\
0 \to & K_m \oplus K'_m & \to & P_{m-1} \oplus P'_{m-1} & \to \cdots \to & P_0 \oplus P'_0 \to & X & \to 0 \\
& \downarrow_{j_m \circ \pi_m} & & \downarrow_{j_{m-1} \circ \pi_{m-1}} & & \downarrow_{j_0 \circ \pi_0} & \downarrow_{e} & \\
0 \to & K_m \oplus K'_m & \to & P_{m-1} \oplus P'_{m-1} & \to \cdots \to & P_0 \oplus P'_0 \to & X & \to 0 \\
& \downarrow_{j \circ h \circ \pi_m} & & & & & & \\
& X & & & & & &
\end{array}$$

By the definition of Yoneda product, we have

$$\begin{aligned}&\overline{j\circ h\circ \pi_m}\cdot e\cdot \overline{j\circ g\circ \pi_n}=\overline{j\circ h\circ \pi_m\circ j_m\circ \pi_m\circ (g_m,0)}\\ =\ &\overline{j\circ h\circ \pi_m\circ (g_m,0)}=\overline{j\circ h\circ g_m\circ \pi_{n+m}}\,.\end{aligned}$$

Hence, $\theta(\overline{h}\cdot\overline{g})=\theta(\overline{h})\cdot\theta(\overline{g})$. This completes the proof. □

Proposition 1.3. *With the notations in Lemma* 1.1, *if* $Y'\in \mathrm{add}(Y)$, *then* $\mathcal{E}(X)$ *and* $\mathcal{E}(Y)$ *are graded equivalent.*

Proof. By Lemma 1.1, $\mathrm{Ext}^*_A(Y,X)\cong \mathcal{E}(X)\cdot e$ as left graded $\mathcal{E}(X)$-modules. Hence, $\mathrm{Ext}^*_A(Y,X)$ is a finitely generated graded projective $\mathcal{E}(X)$-module. Since $Y'\in\mathrm{add}(Y)$, there exists an A-module Z such that $Y'\oplus Z=Y^{\oplus n}$ for some $n\geq 1$. Now $X\oplus Z=Y^{\oplus n+1}$. Hence, we have

$$\begin{aligned}&\mathrm{Ext}^*_A(Y,X)^{\oplus n+1}\cong \mathrm{Ext}^*_A(Y^{\oplus n+1},X)\\ \cong\ &\mathrm{Ext}^*_A(X\oplus Z,X)\cong \mathcal{E}(X)\oplus \mathrm{Ext}^*_A(Z,X)\end{aligned}$$

as graded left $\mathcal{E}(X)$-modules. Hence, the graded module $\mathrm{Ext}^*_A(Y,X)\cong\mathcal{E}(X)\cdot e$ is a finitely generated projective generator of the category $\mathcal{E}(X)$-Mod. By [3, Theorem 5.4], $\mathcal{E}(X)$ is graded equivalent to the graded algebra $\mathrm{End}_{\mathcal{E}(X)}(\mathrm{Ext}^*_A(Y,X))^{\mathrm{op}}\cong \mathrm{End}_{\mathcal{E}(X)}(\mathcal{E}(X)\cdot e)^{\mathrm{op}}\cong e\cdot\mathcal{E}(X)\cdot e$. By Lemma 1.2, $\mathcal{E}(Y)\cong e\cdot\mathcal{E}(X)\cdot e$. Hence, $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are graded equivalent. □

Let A be a semiperfect algebra with Jacobson radical J. Let $\{e_1,\dots,e_n\}$ be a set of complete orthogonal primitive idempotents of A, and let $\{e_{i_1},\dots,e_{i_t}\}$ be a set of basic idempotents. Write $e=e_{i_1}+\cdots+e_{i_t}$. Then Ae is a finitely generated projective generator of the category A-Mod. Let $F=\mathrm{Hom}_A(Ae,-)$. Then F is a Morita equivalence from A-Mod to eAe-Mod. Hence, $\mathrm{Ext}^*_A(A/J,A/J)\cong \mathrm{Ext}^*_{eAe}(F(A/J),F(A/J))$ as graded algebras. On the other hand,

$$\begin{aligned}&F(A/J)=\mathrm{Hom}_A(Ae,A/J)\cong \mathrm{Hom}_A\big(Ae,\textstyle\bigoplus_{i=1}^n(Ae_i/Je_i)\big)\\ \cong\ &\textstyle\bigoplus_{i=1}^n(eAe_i/eJe_i)\cong\bigoplus_{s=1}^t(eAe_{i_s}/eJe_{i_s})\oplus\bigoplus_{s\in\Lambda}(eAe_s/eJe_s),\end{aligned}$$

where $\Lambda=\{e_1,\dots,e_n\}\setminus\{e_{i_1},\dots,e_{i_t}\}$. Set $X=F(A/J)$, $Y=\bigoplus_{s=1}^t(eAe_{i_s}/eJe_{i_s})$ and $Y'=\bigoplus_{s\in\Lambda}(eAe_s/eJe_s)$. Since $\{e_{i_1},\dots,e_{i_t}\}$ is a set of basic idempotents, we have $Y'\in\mathrm{add}(Y)$. By Proposition 1.3, the graded algebras $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are graded equivalent. Clearly, $Y=\bigoplus_{s=1}^t(eAe_{i_s}/eJe_{i_s})=eAe/eJe$. Since eJe is the Jacobson radical of eAe, $\mathrm{Ext}^*_{eAe}(eAe/eJe,eAe/eJe)$ is graded equivalent to $\mathcal{E}(X)\cong\mathrm{Ext}^*_A(A/J,A/J)$. In other words, the Yoneda Ext-algebra of a semiperfect algebra A is graded equivalent to the Yoneda Ext-algebra of its basic algebra.

Let A be a semiperfect algebra with Jacobson radical J. In what follows, we write $E(A)=\mathrm{Ext}^*_A(A/J,A/J)$.

The above argument proves the following main result of this section.

Theorem 1.4. *Let* A *and* A' *be semiperfect algebras. If* A *and* A' *are Morita equivalent, then the graded algebras* $E(A)$ *and* $E(A')$ *are graded equivalent.*

2 Ext-Algebras of Noetherian Semiperfect Algebras

Let A be a noetherian semiperfect algebra with Jacobson radical J. As we know, the Yoneda Ext-algebra $E(A) = \mathrm{Ext}_A^*(A/J, A/J)$ is a positively $\mathbb{Z}$-graded algebra. In this section, we construct a graded right ideal $\overline{E}(A) = \bigoplus_{i\geq 1}\overline{E}^i(A)$ of $E(A)$, which will play an important role in the discussions in the rest of the paper. We introduce an extra grading on $\overline{E}^i(A)$ for all $i \geq 1$ so that $\overline{E}(A)$ is a bigraded space. The extra grading on $\mathrm{Ext}_A^*(A/J, A/J)$ brings us some convenience in the study of the structure of $E(A)$.

We first fix some notations. In the rest of this paper, A is a noetherian semiperfect algebra with Jacobson radical J, and

$$\cdots \to P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A/J \to 0 \tag{1}$$

is a minimal projective resolution of A/J with $P_0 = A$ and d_0 the natural projection. Let $K_i = \mathrm{Ker}(d_{i-1})$ be the i-syzygy of A/J. Let N be a finitely generated left A-module. We sometimes use $\Omega^i(N)$ to denote the i-syzygy of N.

Since (1) is minimal, we have $K_i \subseteq JP_{i-1}$ for all $i \geq 1$. Evidently, K_i/JK_i is a left A/J-module, which admits a natural filtration:

$$\begin{aligned} K_i/JK_i &= (K_i \cap JP_{i-1})/JK_i \supseteq (K_i \cap J^2P_{i-1})/JK_i \\ &\supseteq (K_i \cap J^3P_{i-1} + JK_i)/JK_i \supseteq \cdots \supseteq (K_i \cap J^nP_{i-1} + JK_i)/JK_i \supseteq \cdots. \end{aligned} \tag{2}$$

For $i \geq 1$, write

$$\begin{aligned} W_0^i &= K_i/JK_i = (K_i \cap JP_{i-1})/JK_i, \\ W_1^i &= (K_i \cap J^2P_{i-1})/JK_i, \\ &\cdots \\ W_n^i &= (K_i \cap J^nP_{i-1} + JK_i)/JK_i, \\ &\cdots \end{aligned}$$

Since A is noetherian, K_i is finitely generated for all $i \geq 1$. Hence, K_i/JK_i is a finitely generated A/J-module. Since A/J is semisimple, the filtration (2) satisfies the descending chain condition. For $i \geq 1$, there is a sufficiently large integer N_i such that $W_t^i = W_{t+1}^i$ for all $t \geq N_i$. Let $T^i = \bigcap_{j\geq 0} W_j^i$. Since A/J is semisimple, from the filtration (2), we get $K_i/JK_i \cong \bigoplus_{j\geq 0} W_j^i/W_{j+1}^i \oplus T^i$. Since

$$\begin{aligned} W_0^i/W_1^i &\cong K_i/(K_i \cap J^2P_{i-1}), \\ W_1^i/W_2^i &\cong (K_i \cap J^2P_{i-1})/(JK_i \cap J^2P_{i-1} + K_i \cap J^3P_{i-1}), \\ &\cdots \\ W_n^i/W_{n+1}^i &\cong (K_i \cap J^{n+1}P_{i-1})/(JK_i \cap J^{n+1}P_{i-1} + K_i \cap J^{n+2}P_{i-1}), \\ &\cdots \end{aligned}$$

we get

$$\begin{aligned} K_i/JK_i \cong\ & K_i/(K_i \cap J^2P_{i-1}) \oplus (K_i \cap J^2P_{i-1})/(JK_i \cap J^2P_{i-1} + K_i \cap J^3P_{i-1}) \\ & \oplus \cdots \oplus (K_i \cap J^nP_{i-1})/(JK_i \cap J^nP_{i-1} + K_i \cap J^{n+1}P_{i-1}) \oplus \cdots \oplus T^i. \end{aligned}$$

For simplicity, write

$$\begin{aligned}
H_0^i &= K_i/(K_i \cap J^2 P_{i-1}),\\
H_1^i &= (K_i \cap J^2 P_{i-1})/(JK_i \cap J^2 P_{i-1} + K_i \cap J^3 P_{i-1}),\\
&\cdots\\
H_n^i &= (K_i \cap J^{n+1} P_{i-1})/(JK_i \cap J^{n+1} P_{i-1} + K_i \cap J^{n+2} P_{i-1}),\\
&\cdots
\end{aligned}$$

Since (1) is a minimal resolution, we have

$$\begin{aligned}
&\operatorname{Ext}_A^i(A/J, A/J) \cong \operatorname{Hom}_A(K_i, A/J) \cong \operatorname{Hom}_{A/J}(K_i/JK_i, A/J)\\
\cong\ &\textstyle\bigoplus_{j\ge 0} \operatorname{Hom}_{A/J}(H_j^i, A/J) \oplus \operatorname{Hom}_{A/J}(T^i, A/J).
\end{aligned} \tag{3}$$

In what follows, we identify the terms in (3).

For $i \geq 1$, let $\overline{E}^i(A) = \bigoplus_{j\ge 0} \operatorname{Hom}_{A/J}(H_j^i, A/J)$, and $\overline{E}(A) = \bigoplus_{i\ge 1} \overline{E}^i(A)$. Then $\overline{E}(A)$ is a graded subspace of $E(A)$.

We introduce an extra grading on $\overline{E}^i(A)$ for $i \geq 1$. Let

$$\overline{E}_j^i(A) = \begin{cases} 0 & \text{if } j < i,\\ \operatorname{Hom}_{A/J}(H_{j-i}^i, A/J) & \text{if } j \geq i. \end{cases}$$

We call this the *second grading* of $\overline{E}(A)$. Then $\overline{E}(A)$ is a bigraded space.

Clearly, if the Jacobson radical J of A has the left Artin–Rees property (that is, for any left ideal I of A, $I \cap J^n \subseteq JI$ for sufficiently large n), then for $i \geq 1$, we have $K_i \cap J^{n_i} P_{i-1} \subseteq JK_i$ for sufficiently large n_i. In this case, $T^i = 0$. Hence, $E^i(A) = \overline{E}^i(A)$. If we let $E_0^0(A) = \operatorname{Hom}_{A/J}(A/J, A/J) = A/J$ and $E_j^0(A) = 0$ for $j \neq 0$, then $E(A)$ is a bigraded space. However, in general, we do not know whether J has the Artin–Rees property. Hence, we do not know whether $E^i(A)$ coincides with $\overline{E}^i(A)$.

Lemma 2.1. *For $g \in \overline{E}_t^s(A)$ and $f \in E^i(A)$, $g \cdot f \in \overline{E}_{i+s}^{i+s}(A) \oplus \cdots \oplus \overline{E}_{i+t}^{i+s}(A)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccccccc}
0 \to & K_{i+s} & \xrightarrow{\tau_{i+s}} & P_{i+s-1} & \to \cdots \to & P_i & \to & K_i & \to & P_{i-1} & \to & \cdots \\
& \downarrow f_s & & \downarrow f_{s-1} & & \downarrow f_0 & & \downarrow f & & & & \\
0 \to & K_s & \xrightarrow{\tau_s} & P_{s-1} & \to \cdots \to & P_0 & \to & A/J & \to & 0 & & \\
& \downarrow g & & & & & & & & & & \\
& A/J & & & & & & & & & &
\end{array}$$

where f_j is induced by f for $0 \leq j \leq s$, and τ_{i+s} and τ_s are the inclusions. By definition, $g \cdot f = g \circ f_s$. For $x \in K_{i+s} \cap J^{t-s+l} P_{i+s-1}$ $(l \geq 2)$, we have $\tau_s \circ f_s(x) = f_{s-1} \circ \tau_{i+s}(x) \in J^{t-s+l} P_{s-1}$. Hence, $f_s(x) \in K_s \cap J^{t-s+l} P_{i+s-1}$. Since $g \in E_t^s(A) \cong \operatorname{Hom}_{A/J}(H_{t-s}^i, A/J)$, we get $g \circ f_s(x) = 0$. So $g \cdot f \in \overline{E}_{i+s}^{i+s}(A) \oplus \cdots \oplus \overline{E}_{i+t}^{i+s}(A)$. □

Corollary 2.2. *$E^1(A)=\overline{E}^1(A)$, $E^1(A)\cdot E^{i-1}(A)\subseteq \overline{E}_i^i(A)$, and $\overline{E}_i^i(A)\cdot E^s(A)\subseteq \overline{E}_{i+s}^{i+s}(A)$.*

Proof. From the minimal resolution (1), we get $K_1=J$. Therefore, $H_0^1=J/J^2$, $H_j^1=0$ for $j\geq 1$, and $T^1=0$. Hence, $E^1(A)=\overline{E}^1(A)$. Moreover, $\overline{E}^1(A)$ is concentrated in degree $(1,1)$. By Lemma 2.1, we have $E^1(A)\cdot E^{i-1}(A)\subseteq \overline{E}_i^i(A)$. The rest one follows from Lemma 2.1 directly. □

Corollary 2.3. *$\overline{E}(A)=\bigoplus_{i\geq 1}\overline{E}^i(A)$ is a graded right ideal of $E(A)$.*

Let $R=k\oplus R_1\oplus R_2\oplus\cdots$ be a connected graded algebra. In [8], the subalgebra of $E(R)=\mathrm{Ext}_R^*(k,k)$ generated by $E^1(R)=\mathrm{Ext}_R^1(k,k)$ is described explicitly. In our case, the subalgebra of $E(A)$ generated by $E^1(A)$ is not very clear. However, we give a description of the right ideal of $E(A)$ generated by $E^1(A)$. Set $\widetilde{E}(A)=\bigoplus_{i\geq 1}\overline{E}_i^i(A)$. By Corollary 2.2, we know that $\widetilde{E}(A)$ is a right ideal of $E(A)$. Let $E^1(A)\cdot E(A)$ denote the right ideal of $E(A)$ generated by $E^1(A)$. By Corollary 2.2 again, $E^1(A)\cdot E(A)\subseteq \widetilde{E}(A)$.

Theorem 2.4. *$\widetilde{E}(A)=E^1(A)\cdot E(A)$.*

Proof. It suffices to prove $\overline{E}_i^i(A)=E^1(A)\cdot E^{i-1}(A)$ for all $i\geq 2$. From the minimal resolution (1), we have $K_i\subseteq JP_{i-1}$. Let θ be the composition

$$K_i\overset{\tau}{\hookrightarrow} JP_{i-1}\xrightarrow{\pi} JP_{i-1}/J^2P_{i-1},$$

where τ is the inclusion and π is the natural projection. Then $\mathrm{Ker}(\theta)=K_i\cap J^2P_{i-1}$. Hence, we have a monomorphism $\eta: K_i/(K_i\cap J^2P_{i-1})\to JP_{i-1}/J^2P_{i-1}$ induced by θ. Since $K_i/(K_i\cap J^2P_{i-1})$ and JP_{i-1}/J^2P_{i-1} are A/J-modules, we can regard η as a left A/J-module morphism. Since A/J is semisimple, η is split. There is an A/J-module morphism $\xi: JP_{i-1}/J^2P_{i-1}\to K_i/(K_i\cap J^2P_{i-1})$ such that $\xi\circ\eta=1$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
K_i & \xrightarrow{\tau} & JP_{i-1} & \xrightarrow{\pi} & JP_{i-1}/J^2P_{i-1}\\
\downarrow\psi & & & \eta\nearrow\swarrow\xi & \\
K_i/(K_i\cap J^2P_{i-1}) & & & &
\end{array}$$

where ψ is the natural projection. Taking ξ as an A-module morphism, we let $\varphi=\xi\circ\pi: JP_{i-1}\to K_i/(K_i\cap J^2P_{i-1})$. Then $\psi=\varphi\circ\tau$.

For $g\in\mathrm{Hom}_{A/J}\big(K_i/(K_i\cap J^2P_{i-1}),A/J\big)$, let $\widetilde{g}\in\mathrm{Hom}_A(K_i,A/J)$ be the morphism corresponding to g through the isomorphisms in (3), i.e., $\widetilde{g}=g\circ\psi$. Let $f=g\circ\varphi: JP_{i-1}\to A/J$. Then $f\circ\tau=\widetilde{g}$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0\to & K_i & \to & P_{i-1} & \to & K_{i-1} & \hookrightarrow & P_{i-2} & \to\cdots\\
 & \downarrow\tau & & \downarrow 1 & & \downarrow h & & & \\
0\to & JP_{i-1} & \to & P_{i-1} & \to & P_{i-1}/JP_{i-1} & \to & 0 & \\
 & \downarrow f & & & & & & & \\
 & A/J & & & & & & &
\end{array}$$

where h is the natural morphism induced by τ. We get $\widetilde{g} = f \cdot h$ by definition. Clearly, P_{i-1}/JP_{i-1} is a semisimple A-module. We may assume $P_{i-1}/JP_{i-1} = \bigoplus_{t=1}^{n} S_t$, in which each S_t is a simple A-module. Then we may write $h = (h_1, \ldots, h_n)$ with $h_t \in \mathrm{Ext}_A^{i-1}(A/J, S_t) \subseteq \mathrm{Ext}_A^{i-1}(A/J, A/J)$. Since $f \in \mathrm{Ext}_A^1\big(\bigoplus_{t=1}^{n} S_t, A/J\big)$, we may write $f = (f_1, \ldots, f_n)$ with $f_t \in \mathrm{Ext}_A^1(S_t, A/J) \subseteq \mathrm{Ext}_A^1(A/J, A/J)$. Then $\widetilde{g} = f \cdot h = \sum_{t=1}^{n} f_t \cdot h_t$. Hence, $g \in E^1(A) \cdot E^{i-1}(A)$. □

3 Ext-Algebras of (Higher) Quasi-Koszul Algebras

Let A be a noetherian semiperfect algebra with Jacobson radical J. A is called a *quasi-Koszul algebra* (see [5]) if the minimal resolution (1) of A/J has the property

$$K_i \cap J^2 P_{i-1} = JK_i \quad \text{for all } i \geq 1. \tag{4}$$

A is called a *higher quasi-Koszul algebra* if for the minimal resolution (1) of A/J, there is an integer $p \geq 2$ such that

(i) if i is odd, then $K_i \cap J^2 P_{i-1} = JK_i$; and

(ii) if i is even, then $K_i \subseteq J^{p-1} P_{i-1}$ and $K_i \cap J^p P_{i-1} = JK_i$.

More indicatively, we call a higher quasi-Koszul algebra as a *quasi-p-Koszul algebra*. Clearly, a quasi-2-Koszul algebra is a usual quasi-Koszul algebra.

If the noetherian algebra A is positively graded and the Jacobson radical J is replaced by the graded Jacobson radical, then our definition of quasi-Koszul algebras coincides with that of higher Koszul algebras given in [1] and [4].

The following two lemmas are obvious.

Lemma 3.1. *If A is a quasi-p-Koszul algebra and ${}_AS$ is a finitely generated semisimple A-module, then ${}_AS$ has a minimal projective resolution $\cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to S \to 0$ such that*

(i) *$\Omega^i(S) \cap J^2 Q_{i-1} = J\Omega^i(S)$ when i is odd*; *and*

(ii) *$\Omega^i(S) \subseteq J^{p-1} Q_{i-1}$ and $\Omega^i(S) \cap J^p Q_{i-1} = J\Omega^i(S)$ when i is even.*

Lemma 3.2. *Let A be a noetherian semiperfect algebra. For a given integer $i \geq 2$, if $K_i \subseteq J^t P_{i-1}$ for $t \geq 1$, then for any finitely generated semisimple A-module ${}_AS$, there is a minimal projective resolution $\cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to S \to 0$ such that $\Omega^i(S) \subseteq J^t Q_{i-1}$.*

Example 3.3. (i) The formal power series algebra $A = k[[x]]$ is a quasi-Koszul algebra. In fact, the residue field k has a minimal resolution $0 \to A \xrightarrow{x} A \to k \to 0$, which certainly satisfies (4). This is a special case of the following more general one.

(ii) Let A be a regular local algebra (see [10]). Then A is a quasi-Koszul algebra. In fact, let $\mathfrak{m}$ be the maximal ideal of A, and assume $G(A) = n$. Let $x_1, \ldots, x_n$ be a regular sequence of A. Then the residue field $A/\mathfrak{m}$ has a minimal resolution

$$0 \to \wedge^n(A^{\oplus n}) \xrightarrow{d_n} \cdots \to \wedge^t(A^{\oplus n}) \xrightarrow{d_t} \cdots \to \wedge^1(A^{\oplus n}) \xrightarrow{d_1} A \to A/\mathfrak{m} \to 0,$$

where $d_t : \wedge^t(A^{\oplus n}) \to \wedge^{t-1}(A^{\oplus n})$ is defined as follows. Let $e_1, \ldots, e_n$ be a free basis of $A^{\oplus n}$. Then $d_t(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{s=1}^{t} (-1)^{s-1} x_s e_{i_1} \wedge \cdots \wedge \hat{e}_{i_s} \wedge \cdots \wedge e_{i_t}$, in which

$\hat{e}_{i_s}$ means that e_{i_s} is deleted. It is not hard to see that this minimal resolution of $A/\mathfrak{m}$ satisfies the identity (4). Hence, A is a quasi-Koszul algebra.

Example 3.4. Let S be the commutative ring $k^{\oplus 5}$, and let $\{e_1, \cdots, e_5\}$ be the set of primitive orthogonal idempotents of S. Let $V = kv_1 \oplus kv_2 \oplus kv_3 \oplus kv_4$ be a vector space of dimension 4. Define an S-S-bimodule action on V as follows. For $1 \le i \le 5$ and $1 \le j \le 4$, let

$$e_i \cdot v_j = \begin{cases} 0 & \text{if } j \neq i-1, \\ v_j & \text{if } j = i-1, \end{cases} \qquad v_j \cdot e_i = \begin{cases} 0 & \text{if } j \neq i, \\ v_j & \text{if } j = i. \end{cases}$$

It is not hard to check that V is an S-S-bimodule. Let $T(V) = S \oplus V \oplus V \otimes_S V \oplus \cdots$. Then $T(V)$ is a k-algebra. Let I be the ideal of $T(V)$ generated by $v_4 \otimes_S v_3 \otimes_S v_2$, and let $A = T(V)/I$. Then A is a noetherian semiperfect algebra. One can easily check that A is a quasi-3-Koszul algebra.

Quasi-p-Koszul algebras are characterized by the second grading of $\overline{E}(A)$ introduced in Section 2.

Proposition 3.5. *Let A be a noetherian semiperfect algebra. Then A is a quasi-p-Koszul algebra if and only if for all $i \ge 1$,*

(i) $E^i(A) = \overline{E}^i(A)$;

(ii) $\overline{E}^i(A)$ *is concentrated in degree (i, i) when i is odd; and*

(iii) $\overline{E}^i(A)$ *is concentrated in degree $(i,\ i+p-2)$ when i is even.*

Proof. Assume that A is a quasi-p-Koszul algebra. By definition, $K_i \cap J^p P_{i-1} \subseteq JK_i$ for all $i \ge 1$. Hence, $T^i = 0$ for all $i \ge 1$. Then $E^i(A) = \overline{E}^i(A)$ for all $i \ge 1$. Let i be odd. Then $K_i \cap J^2 P_{i-1} = JK_i$. Hence, $K_i \cap J^s P_{i-1} \subseteq JK_i$ for all $s \ge 2$. Then $K_i \cap J^s P_{i-1} = JK_i \cap J^s P_{i-1}$ for all $s \ge 2$. Hence, for $t \ge 1$,

$$H_t^i = (K_i \cap J^{t+1} P_{i-1})/(JK_i \cap J^{t+1} P_{i-1} + K_i \cap J^{t+2} P_{i-1}) = 0.$$

Then $\overline{E}_j^i(A) = \mathrm{Hom}_{A/J}(H_{j-i}^i, A/J) = 0$ for all $j - i \ge 1$; i.e., $\overline{E}^i(A)$ is concentrated in degree (i, i).

Now let i be even. Then $K_i \subseteq J^{p-1} P_{i-1}$ and $K_i \cap J^p P_{i-1} = JK_i$. Hence, $K_i \cap J^s P_{i-1} = K_i \cap J^{s+1} P_{i-1} = K_i$ for all $1 \le s \le p-2$. We then have

$$H_t^i = (K_i \cap J^{t+1} P_{i-1})/(JK_i \cap J^{t+1} P_{i-1} + K_i \cap J^{t+2} P_{i-1}) = K_i/K_i = 0$$

for all $1 \le t \le p-3$. Hence, $\overline{E}_j^i(A) = \mathrm{Hom}_{A/J}(H_{j-i}^i, A/J) = 0$ for all $1 \le j - i \le p-3$; i.e., $\overline{E}_i^i(A) = \cdots = \overline{E}_{i+p-3}^i(A) = 0$. On the other hand, $K_i \cap J^s P_{i-1} = JK_i \cap J^s P_{i-1}$ for $s \ge p$ since $K_i \cap J^p P_{i-1} = JK_i$. Hence, we also have

$$H_t^i = (K_i \cap J^{t+1} P_{i-1})/(JK_i \cap J^{t+1} P_{i-1} + K_i \cap J^{t+2} P_{i-1}) = 0$$

for $t \ge p-1$; i.e., $\overline{E}_j^i(A) = 0$ for $j \ge i+p-1$. Therefore, $\overline{E}^i(A)$ is concentrated in degree $(i,\ i+p-2)$ when i is even.

Conversely, assume that $E^i(A)$ has the properties as stated in the proposition. If i is odd, then $\overline{E}^i_j(A) = \mathrm{Hom}_{A/J}(H^i_{j-i}, A/J) = 0$ for $j - i \geq 1$. Since A/J is semisimple, we get $H^i_{j-i} = 0$ for $j - i \geq 1$. Hence,

$$(K_i \cap J^{t+1}P_{i-1})/(JK_i \cap J^{t+1}P_{i-1} + K_i \cap J^{t+2}P_{i-1}) = 0$$

for all $t \geq 1$. Since $E^i(A) = \overline{E}^i(A)$, we get $T^i = 0$. So there is an integer N (depending on i) such that $K_i \cap J^N P_{i-1} \subseteq JK_i$. Then $K_i \cap J^N P_{i-1} = JK_i \cap J^N P_{i-1}$. Hence, $K_i \cap J^{N-1}P_{i-1} = JK_i \cap J^{N-1}P_{i-1} + K_i \cap J^N P_{i-1} = JK_i \cap J^{N-1}P_{i-1}$, and

$$\begin{aligned} K_i \cap J^{N-2}P_{i-1} &= JK_i \cap J^{N-2}P_{i-1} + K_i \cap J^{N-1}P_{i-1} \\ &= JK_i \cap J^{N-2}P_{i-1} + JK_i \cap J^{N-1}P_{i-1} = JK_i \cap J^{N-2}P_{i-1}. \end{aligned}$$

Inductively, we obtain $K_i \cap J^s P_{i-1} = JK_i \cap J^s P_{i-1}$ for all $s \geq 2$. In particular, $K_i \cap J^2 P_{i-1} = JK_i \cap J^2 P_{i-1} \subseteq JK_i$. Obviously, $JK_i \subseteq K_i \cap J^2 P_{i-1}$. Hence, $JK_i = K_i \cap J^2 P_{i-1}$ if i is odd.

Now let i be even. Then $H^i_t = 0$ for $t \neq p-2$. Then $H^i_0 = K_i/(K_i \cap J^2 P_{i-1}) = 0$ implies $K_i = K_i \cap J^2 P_{i-1}$. Hence, $K_i \subseteq J^2 P_{i-1}$. Similarly, $H^i_1 = 0$ implies $K_i \cap J^2 P_{i-1} = JK_i \cap J^2 P_{i-1} + K_i \cap J^3 P_{i-1}$, which implies $K_i = JK_i + K_i \cap J^3 P_{i-1}$. Since K_i is finitely generated, JK_i is a superfluous submodule of K_i. Hence, $K_i = K_i \cap J^3 P_{i-1}$, which implies $K_i \subseteq J^3 P_{i-1}$. Inductively, we can show $K_i \subseteq J^t P_{i-1}$ for $2 \leq t \leq p-1$. The identity $K_i \cap J^p P_{i-1} = JK_i$ follows from the similar analysis as in the case that i is odd. Therefore, $K_i \subseteq J^{p-1}P_{i-1}$ and $K_i \cap J^p P_{i-1} = JK_i$ when i is even. This completes the proof. □

From the proof of Proposition 3.5, we have:

Corollary 3.6. *Let A be a noetherian semiperfect algebra and $i \geq 2$.*

(i) *If $K_i \subseteq J^t P_{i-1}$, then $E^i_j(A) = 0$ for $j < i + t - 1$.*

(ii) *$E^i(A) = \overline{E}^i(A)$, and $E^i(A)$ is concentrated in degree $(i, i+t)$ for some $t \geq 0$ if and only if $K_i \subseteq J^{t+1}P_{i-1}$ and $K_i \cap J^{t+2}P_{i-1} = JK_i$.*

As a consequence of Proposition 3.5, we give a short proof of a result of Green and Martínez Villa:

Theorem 3.7. [5, Theorem 4.4] *Let A be a notherian semiperfect algebra. Then A is a quasi-Koszul algebra if and only if $E(A)$ is generated by $E^1(A)$.*

Proof. Suppose that A is a quasi-Koszul algebra. Then $E^i(A) = \overline{E}^i(A)$ and $\overline{E}^i(A)$ is concentrated in degree (i, i) for all $i \geq 1$ by Proposition 3.5. By Theorem 2.4, $E^i(A) = \overline{E}^i_i(A) = E^1(A) \cdot E^{i-1}(A)$ for all $i \geq 2$; i.e., $E(A)$ is generated by $E^1(A)$.

Conversely, if $E(A)$ is generated by $E^1(A)$, then $E^i(A) = E^1(A) \cdot E^{i-1}(A)$. But Corollary 2.2 says that $E^1(A) \cdot E^{i-1}(A) \subseteq \overline{E}^i_i(A)$. Hence, $E^i(A) = \overline{E}^i_i(A)$; i.e., $E^i(A) = \overline{E}^i(A)$ and $\overline{E}^i(A)$ is concentrated in degree (i, i) for all $i \geq 1$. By Proposition 3.5, A is a quasi-Koszul algebra. □

We next show that there is a result similar to Theorem 3.7 for higher quasi-Koszul algebras.

Proposition 3.8. *Let A be a quasi-p-Koszul algebra $(p \geq 3)$. Then for all $i \geq 0$, $E^{2i+1}(A) \cdot E^{2j+1}(A) = 0$.*

Proof. Since A is a quasi-p-Koszul algebra, $E^{2i+1}(A) = \overline{E}^{2i+1}(A)$ and $\overline{E}^{2i+1}(A)$ is concentrated in degree $(2i+1, 2i+1)$ by Proposition 3.5. Then $E^{2i+1}(A) \cdot E^{2j+1}(A) \subseteq \overline{E}^{2(i+j)+2}_{2(i+j)+2}(A)$. But by Proposition 3.5 again, $\overline{E}^{2(i+j)+2}(A)$ is concentrated in degree $(2(i+j)+2,\ 2(i+j)+p)$. Since $p \geq 3$, $2(i+j)+p > 2(i+j)+2$ and we get $E^{2i+1}(A) \cdot E^{2j+1}(A) = 0$. □

Let A be a noetherian semiperfect algebra. Set $E^{\mathrm{ev}}(A) = \bigoplus_{i \geq 0} E^{2i}(A)$ and $E^{\mathrm{odd}}(A) = \bigoplus_{i \geq 0} E^{2i+1}(A)$. Then $E^{\mathrm{ev}}(A)$ is a subalgebra of $E(A)$, and $E^{\mathrm{odd}}(A)$ is a right $E^{\mathrm{ev}}(A)$-module. If A is a quasi-p-Koszul algebra with $p \geq 3$, then $E^{\mathrm{odd}}(A)$ is also a right $E(A)$-module by Proposition 3.8.

Lemma 3.9. *Let A be a noetherian semiperfect algebra. Then A is a quasi-p-Koszul algebra if*

(i) *$K_2 \subseteq J^{p-1}P_1$ and $K_2 \cap J^p P_1 = JK_2$;*
(ii) *the subalgebra $E^{\mathrm{ev}}(A)$ is generated by $E^0(A)$ and $E^2(A)$; and*
(iii) *the right $E^{\mathrm{ev}}(A)$-module $E^{\mathrm{odd}}(A)$ is generated by $E^1(A)$.*

Proof. By (iii), we have $E^{2i+1}(A) = E^1(A) \cdot E^{2i}(A)$ for all $i \geq 1$. By Corollary 2.2, we have $E^1(A) \cdot E^{2i}(A) \subseteq \overline{E}^{2i+1}_{2i+1}(A)$. Hence, $E^{2i+1}(A) = \overline{E}^{2i+1}(A)$ and $\overline{E}^{2i+1}(A)$ is concentrated in degree $(2i+1, 2i+1)$. We then get $K_{2i+1} \subseteq JP_{2i}$ and $K_{2i+1} \cap J^2 P_{2i} = JK_{2i+1}$ by Corollary 3.6.

Next we show $K_{2i+2} \subseteq J^{p-1}P_{2i+1}$ for all $i \geq 1$. Since $P_{2i} \xrightarrow{d_{2i}} K_{2i} \to 0$ is a projective cover, we have short exact sequences $0 \to JP_{2i} \to P_{2i} \to K_{2i}/JK_{2i} \to 0$ and $0 \to K_{2i+1} \to JP_{2i} \to JK_{2i} \to 0$. Since $K_{2i+1} \cap J^2 P_{2i} = JK_{2i+1}$, we have the following exact commutative diagram with $Q \to JP_{2i} \to 0$ and $Q' \to JK_{2i} \to 0$ projective covers:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_{2i+2} & \xrightarrow{\theta} & \Omega^2(K_{2i}/JK_{2i}) & \to & K' & \to & 0 \\
 & & \downarrow \tau & & \downarrow \tau'' & & \downarrow \tau' & & \\
0 & \to & P_{2i+1} & \xrightarrow{\varphi} & Q & \to & Q' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_{2i+1} & \xrightarrow{\eta} & JP_{2i} & \to & JK_{2i} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (5)$$

where τ, τ', τ'' are the inclusions. Since K_{2i}/JK_{2i} is a finitely generated semisimple A-module, we get $\Omega^2(K_{2i}/JK_{2i}) \subseteq J^{p-1}Q$ by Lemma 3.2. Hence, $\tau'' \circ \theta(K_{2i+2}) \subseteq J^{p-1}Q$. Since the middle row of the above diagram is split, there is a morphism $\xi : Q \to P_{2i+1}$ such that $\xi \circ \varphi = 1$. Then $\tau(K_{2k+2}) = \xi \circ \tau'' \circ \theta(K_{2k+2}) \subseteq \xi(J^{p-1}Q) \subseteq J^{p-1}P_{2i+1}$; that is, $K_{2i+2} \subseteq J^{p-1}P_{2i+1}$. Hence,

$$\overline{E}^{2i}_j(A) = 0 \quad \text{for all } j < 2i + p - 2 \qquad (6)$$

by Corollary 3.6. Since $K_2 \subseteq J^{p-1}P_1$ and $K_2 \cap J^pP_1 = JK_2$, we have $E^2(A) = \overline{E}^2(A)$, and $\overline{E}^2(A)$ is concentrated in degree $(2,p)$ by Corollary 3.6. Then from the hypothesis (ii) and Lemma 2.1, we have

$$E^{2i}(A) = E^2(A) \cdot E^{2i-2}(A) = \overline{E}^2(A) \cdot E^{2i-2}(A) \subseteq \overline{E}^{2i}_{2i}(A) \oplus \cdots \oplus \overline{E}^{2i}_{2i+p-2}(A).$$

Combining with (6), we have that $E^{2i}(A) = \overline{E}^{2i}(A)$, and $\overline{E}^{2i}(A)$ is concentrated on $\overline{E}^{2i}_{2i+p-2}(A)$ for all $i \geq 2$.

By Proposition 3.5, we get that A is a quasi-p-Koszul algebra. □

We next show that the converse of the above lemma is also true. At first we prove the following lemma.

Lemma 3.10. *Let A be a quasi-p-Koszul algebra. Then we also have the commutative diagram* (5). *Moreover, $\theta(K_{2i+2}) \cap J\Omega^2(K_{2i}/JK_{2i}) = J\theta(K_{2i+2})$.*

Proof. Since A is a quasi-p-Koszul algebra, $K_{2i+1} \subseteq JP_{2i}$ and $K_{2i+1} \cap J^2P_{2i} = JK_{2i+1}$. Then we have the commutative diagram (5). For simplicity, write $L = \Omega^2(K_{2i}/JK_{2i})$. The morphism $\theta : K_{2i+2} \to L$ induces a morphism $\overline{\theta} : K_{2i+2}/JK_{2i+2} \to L/JL$ in an obvious way. Applying $A/J \otimes_A -$ to the split exact sequence $0 \to P_{2i} \xrightarrow{\varphi} Q \to Q' \to 0$, we get an exact sequence

$$0 \to A/J^p \otimes_A P_{2i+1} \to A/J^p \otimes_A Q \to A/J^p \otimes_A Q' \to 0,$$

which is isomorphic to $0 \to P_{2i+1}/J^pP_{2i+1} \xrightarrow{\overline{\varphi}} Q/J^pQ \to Q'/J^pQ' \to 0$, where $\overline{\varphi}$ is induced from φ in an obvious way. Since $K_{2i+2} \subseteq J^{p-1}P_{2i+1}$, we have $JK_{2i+2} \subseteq J^pP_{2i+1}$. Then the inclusion $K_{2i+2} \hookrightarrow P_{2i+1}$ induces a morphism $\alpha : K_{2i+2}/JK_{2i+2} \to P_{2i+1}/J^pP_{2i+1}$. Since $K_{2i+2} \cap J^pP_{2i+1} = JK_{2i+1}$, α is a monomorphism. Since K_{2i}/JK_{2i} is a finitely generated semisimple A-module, we have $L \subseteq J^{p-1}Q$ and $JL = J^pQ \cap L$ by Lemma 3.1. Similarly, we have a monomorphism $\beta : L/JL \to Q/J^pQ$. Now we have the following diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & K_{2i+2}/JK_{2i+2} & \xrightarrow{\overline{\theta}} & L/JL \\
 & & \downarrow \alpha & & \downarrow \beta \\
0 & \to & P_{2i+1}/J^pP_{2i+1} & \xrightarrow{\overline{\varphi}} & Q/J^pQ
\end{array}$$

It is not hard to see that the diagram above commutes. This forces $\overline{\theta}$ to be a monomorphism, equivalently, $\theta(K_{2i+2}) \cap J\Omega^2(K_{2i}/JK_{2i}) = J\theta(K_{2i+2})$. □

Lemma 3.11. *If A is a quasi-p-Koszul algebra, then the subalgebra $E^{\mathrm{ev}}(A)$ is generated by $E^0(A)$ and $E^2(A)$.*

Proof. It suffices to show $E^{2i+2}(A) = E^2(A) \cdot E^{2i}(A)$ for all $i \geq 1$. With the same notations as in (5), we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 \to & K_{2i+2} & \hookrightarrow & P_{2i+1} & \to & K_{2i+1} & \hookrightarrow & P_{2i} & \to & K_{2i} & \hookrightarrow P_{2i-1} \to \cdots \\
& \downarrow \theta & & \downarrow \varphi & & \downarrow \eta & & \downarrow 1 & & \downarrow \pi & \\
0 \to & \Omega^2(K_{2i}/JK_{2i}) & \to & Q & \to & JP_{2i} & \hookrightarrow & P_{2i} & \to & K_{2i}/JK_{2i} & \to 0
\end{array}$$

where π is the natural projection. Let $f \in \mathrm{Hom}_A(K_{2i+2}, A/J)$. Then there is a morphism $g : K_{2i+2}/JK_{2i+2} \to A/J$ such that $f = g \circ \pi'$ with π' the natural projection $K_{2i+2} \to K_{2i+2}/JK_{2i+2}$. As before, write $L = \Omega^2(K_{2i}/JK_{2i})$. We have the following commutative diagram

$$\begin{array}{ccccc}
K_{2i+2} & = & K_{2i+2} & \xrightarrow{\theta} & L \\
\Big\downarrow f & & \Big\downarrow \pi' & & \Big\downarrow \pi'' \\
A/J & \xleftarrow{g} & K_{2i+2}/JK_{2i+2} & \xrightarrow{\overline{\theta}} & L/JL
\end{array}$$

where π'' is the natural projection and $\overline{\theta}$ is the morphism induced by θ in an obvious way. Since K_{2i+2}/JK_{2i+2} and L/JL are A/J-modules, we may take g and $\overline{\theta}$ as A/J-module morphisms. By Lemma 3.10, $\overline{\theta}$ is a monomorphism. Then we have an A/J-module morphism $\zeta : L/JL \to K_{2i+2}/JK_{2i+2}$ such that $\zeta \circ \overline{\theta} = 1$ since A/J is semisimple. Taking ζ as an A-module morphism, we have $f = g \circ \pi' = g \circ \zeta \circ \pi'' \circ \theta$. Set $h = g \circ \zeta \circ \pi'' : L = \Omega^2(K_{2i}/JK_{2i}) \to A/J$. By identifying $E^{2i+2}(A)$, $\mathrm{Ext}^2_A(K_{2i+2}/JK_{2i+2}, A/J)$ and $\mathrm{Ext}^{2i}_A(A/J, K_{2i}/JK_{2i})$ with $\mathrm{Hom}_A(K_{2i+2}, A/J)$, $\mathrm{Hom}_A(\Omega^2(K_{2i+2}/JK_{2i+2}), A/J)$ and $\mathrm{Hom}_A(K_{2i}, K_{2i}/JK_{2i})$, respectively, we have $f = h \cdot \pi$ by the definition of Yoneda product. Since K_{2i}/JK_{2i} is a finitely generated semisimple A-module, we may assume $K_{2i}/JK_{2i} = \bigoplus_{t=1}^n S_t$ with each S_t a simple A-module. Then we may write $h = (h_1, \ldots, h_n)$ with $h_t \in \mathrm{Ext}^2_A(S_t, A/J) \subseteq E^2(A)$, and $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_t \in \mathrm{Ext}^{2i}_A(A/J, S_t) \subseteq E^{2i}(A)$. Now we have $f = \sum_{t=1}^n h_t \cdot \pi_t \in E^2(A) \cdot E^{2i}(A)$. Hence, $E^{2i+2}(A) = E^2(A) \cdot E^{2i}(A)$ for all $i \geq 1$ as required. □

Now we are ready to prove the following theorem.

Theorem 3.12. *Let A be a noetherian semiperfect algebra, and let $p \geq 3$ be an integer. Then A is a quasi-p-Koszul algebra if and only if*

(i) $K_2 \subseteq J^{p-1}P_1$ *and* $K_2 \cap J^p P_1 = JK_2$;
(ii) *the subalgebra* $E^{\mathrm{ev}}(A)$ *is generated by* $E^0(A)$ *and* $E^2(A)$; *and*
(iii) *the right* $E^{\mathrm{ev}}(A)$*-module* $E^{\mathrm{odd}}(A)$ *is generated by* $E^1(A)$.

Proof. The sufficiency is just Lemma 3.9. Now suppose that A is a quasi-p-Koszul algebra. The condition (i) follows from the definition of quasi-p-Koszul algebras. The condition (ii) is Lemma 3.11. Hence, we only need to prove (iii). By Theorem 2.4, $\widetilde{E}(A) = E^1(A) \cdot E(A)$. But $\widetilde{E}(A) = E^{\mathrm{odd}}(A)$ since $E^{2i}(A) = \overline{E}^{2i}(A)$, and $\overline{E}^{2i}(A)$ is concentrated in degree $(2i,\ 2i+p-2)$ and $2i+p-2 > 2i$ for all $i \geq 1$. By Proposition 3.8, $E^1(A) \cdot E^{2i+1}(A) = 0$ for all $i \geq 0$. Hence, $E^{\mathrm{odd}}(A) = E^1(A) \cdot E(A) = E^1(A) \cdot E^{\mathrm{ev}}(A)$. □

The necessity of the theorem above can be stated in a stronger form.

Corollary 3.13. *If A is a quasi-p-Koszul algebra, then the Yoneda algebra $E(A)$ is generated by $E^0(A)$, $E^1(A)$ and $E^2(A)$.*

4 Quasi-Koszulity is a Morita Invariant

This short section is aimed to show that the quasi-Koszulity of noetherian semiperfect algebras is an invariant under the Morita equivalence.

Lemma 4.1. *Let E and E' be positively graded algebra. If ${}_EP$ is a finitely generated graded progenerator of E-Mod with $E' \cong \mathrm{End}_E(P)^{\mathrm{op}}$, then ${}_EP$ is a positively graded E-module and is generated in degree 0.*

Proof. Let $F = \mathrm{Hom}_E(P, -) : E\text{-GrMod} \to E'\text{-GrMod}$. Then F is a graded equivalence. Assume that the inverse equivalence of F is $G : E'\text{-GrMod} \to E\text{-GrMod}$. Evidently, ${}_EP \cong G(E')$. Since ${}_EP$ is finitely generated, there is an epimorphism of graded modules $\pi : \bigoplus_{i=1}^m S^{n_i}(E) \to {}_EP$ for some integer $m \geq 1$, where S is the shift functor. Then

$$\bigoplus_{i=1}^{m} S^{n_i}(F(E)) = \bigoplus_{i=1}^{m} F(S^{n_i}(E)) \xrightarrow{F\pi} F({}_EP)$$

is a graded epimorphism. Since $F({}_EP) \cong E'$ and E' is generated in degree 0 as a graded module, it follows that there is a graded epimorphism $f : \bigoplus_{j=1}^t S^{n_{i_j}}(F(E)) \to E'$, where $\{n_{i_1}, \ldots, n_{i_t}\}$ is a subset of $\{n_1, \ldots, n_m\}$ such that $n_{i_j} = 0$ for all $1 \leq j \leq t$. In other words, $f : F(E)^{\oplus t} \to E'$ is an epimorphism. Now $GF(E)^{\oplus t} \xrightarrow{G(f)} G(E') \cong {}_EP$ is an epimorphism. Hence, ${}_EP$ is generated in degree 0. □

Corollary 4.2. *Let E and E' be positively graded algebras. If E and E' are graded equivalent, then $E' \cong eM_n(E)e$ for some integer $n \geq 1$ and a homogeneous idempotent $e \in M_n(E)$ of degree 0.*

Proof. Since E and E' are graded equivalent, there is a finitely generated graded progenerator ${}_EP$ with $E' \cong \mathrm{End}_E(P)^{\mathrm{op}}$ as graded algebras. By Lemma 4.1, ${}_EP$ is generated in degree 0. Then there is a graded projective A-module ${}_EP'$ and an integer $n \geq 1$ with $P \oplus P' \cong E^{\oplus n}$. Let $e \in \mathrm{End}_E(E^{\oplus n})^{\mathrm{op}}$ be the idempotent corresponding to P. Clearly, e is a homogeneous element in the graded algebra $\mathrm{End}_E(E^{\oplus n})^{\mathrm{op}}$ of degree 0. Now $E' \cong \mathrm{End}_E(P)^{\mathrm{op}} \cong e\mathrm{End}_E(E^{\oplus n})^{\mathrm{op}}e \cong eM_n(E)e$. This completes the proof. □

Lemma 4.3. *Suppose that the positively graded algebras E and E' are graded equivalent. If E is generated in degrees 0 and 1, then so is E'.*

Proof. By Corollary 4.2, $E' \cong eM_n(E)e$ for some integer n and the idempotent e is of degree 0. Write $B = M_n(E)$. We have $BeB = B$. Since E is generated in degrees 0 and 1, B is also generated in degrees 0 and 1. Then $B_i = B_sB_t$ for all $i \geq 2$ with $s + t = i$ $(s, t \geq 1)$. Hence, for any $x \in B_i$, we have $x = \sum_{j=1}^m y_jz_j$ with $y_j \in B_s$ and $z_j \in B_t$. Then $exe = \sum_{j=1}^m (ey_jz_je)$. Since $BeB = B$, we have

$y_j = \sum_{p=1}^{k_j} u_{jp}ev_{jp}$ and $z_j = \sum_{q=1}^{l_j} w_{jq}eb_{jq}$ with $|u_{jp}| + |v_{jp}| = s$, $|w_{jq}| + |b_{jq}| = t$, and $|u_{jp}|$, $|v_{jp}|$, $|w_{jq}|$, $|b_{jq}| \geq 1$. Now we get

$$\begin{aligned} exe &= \textstyle\sum_{j=1}^{m} e\big(\sum_{p=1}^{k_j} u_{jp}ev_{jp}\big)\big(\sum_{q=1}^{l_j} w_{jq}eb_{jq}\big)e \\ &= \textstyle\sum_{j=1}^{m}\sum_{p=1}^{k_j}\sum_{q=1}^{l_j}(eu_{jp}e)(ev_{jp}w_{jq}e)(eb_{jq}e). \end{aligned}$$

Since $|e| = 0$, it follows that $ev_{jp}w_{jq}e < |x|$. Hence, eBe is generated in degrees 0 and 1; that is, E' is generated in degrees 0 and 1. □

We conclude this section with the following theorem, which shows that the Koszulity is an invariant under Morita equivalence.

Theorem 4.4. *Suppose that the noetherian semiperfect algebras A and A' are Morita equivalent. Then A is quasi-Koszul if and only if A' is quasi-Koszul.*

Proof. It suffices to prove one direction. Suppose that A is a quasi-Koszul algebra. Then $E(A)$ is a positively graded algebra generated in degrees 0 and 1 by Theorem 3.7. Since A and A' are graded equivalent, it follows from Theorem 1.4 that $E(A)$ and $E(A')$ are graded equivalent. By Lemma 4.3, the positively graded algebra $E(A')$ is generated in degrees 0 and 1. Then by Theorem 3.7 again, A' is a quasi-Koszul algebra. □

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